Set theory conditions for stability of linear impulsive systems

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Abstract

In this paper we give tractable necessary and sufficient condition for the global exponential stability of a linear impulsive system. The reset rule considered in the paper is quasi-periodic and the stability analysis is based on a standard tool in set theory that is Minkowski functional. Firstly, we reformulate the problem in term of discrete-time parametric uncertain system with the state matrix belonging to a compact but non-convex set. Secondly, we provide a tractable algorithm for testing the stability and computing the associated polyhedral Lyapunov function when the system is stable. The main result is an algorithm whose computational effort is analogous to that of classical algorithms for contractive polytopes computation for discrete-time parametric uncertain systems with the state matrix belonging to a polytopic set.

This is just a report meant to briefly present our results. For a full version of this work, containing proofs, discussions and numerical implementations please contact the authors.

0.1 Introduction

In order to overcome performance limitations of classical controllers Clegg introduced an integrator with state reset (see [1]). This idea received an increasing attention and recent works have been dedicated to stability and performances of reset control systems [2, 3]. These systems are a class of hybrid systems since they are subject to both continuous-time and discrete-time dynamics. A particular class of reset systems is the continuous-time linear systems whose state undergoes finite jumps at some discrete-time instants [4, 5], also referred to as impulsive systems. The rule defining the jump instants is often time-depending (see [6] and the reference therein) and is motivated by the analysis of sampled-data systems ([7]) as well as periodic triggered stabilization ([8, 9]).

The present paper deals with the stability analysis of linear impulsive systems by means of set theoretic techniques. As in [6], we consider that two consecutive reset instants are separated by an uncertain time. Instead of searching ellipsoidal Lyapunov functions that give sufficient condition for stability, we are searching polyhedral ones leading (as explained later) to necessary and sufficient stability conditions. The stability analysis is based on a standard tool in set theory that is Minkowski functional. Our concern is also to design an algorithm that is able to decide in finite time if a linear impulsive system is globally exponentially stable (GES) or not. In the former case it will also compute in finite time the polyhedral Lyapunov function guaranteeing the stability of the system.

Firstly, we reformulate the problem in term of discrete-time parametric uncertain system with the state matrix belonging to a compact but non-convex set. Secondly, we provide a tractable algorithm for testing the stability and computing the associated polyhedral Lyapunov function when the system is stable. The result is an algorithm whose computational effort is analogous to that of the standard algorithm for computing contractive polytopes for discrete-time polytopic parametric uncertain systems.

Notation. The set of real numbers is denoted by $\mathbb{R}$ while $\mathbb{N}$ stands for the set of positive integer numbers. We denote $\mathbb{N}_n \triangleq \{i \in \mathbb{N},\ i \leq n\}$. For any function x defined on $\mathbb{R}$ we denote $x(t^+) \triangleq \lim\limits_{\tau \mapsto t, \tau > t} x(\tau)$ if the limit exists. A C-set is a convex and compact set containing the origin in its interior. For any real λ and any set S we define $\lambda S \triangleq \{\lambda x \mid x \in S\}$. The unitary ball in $\mathbb{R}^n$ with respect to norm $\|\cdot\|_p$ is $\mathbf{B}_p^n \triangleq \{x \in \mathbb{R}^n :\ \|x\|_p \leq 1\}$, its analogous in the space of matrices is defined in the following.

0.2 Set-theory for nearly-periodic reset systems

Given the interval $\Delta = [\tau_m, \tau_M]$ with $0 < \tau_m < \tau_M \in \mathbb{R}$, we define the set of admissible reset sequences as

$$\Theta(\Delta) = \Big\{\{t_k\}_{k\in\mathbb{N}} :\ t_{k+1} = t_k + \delta_k,\ \delta_k \in \Delta,\ \forall k \in \mathbb{N}\Big\}. \tag{1}$$

The aim of this paper is to give tractable necessary and sufficient conditions for the stability of the following linear reset system

$$\begin{cases} \dot{x}(t) = A_c x(t), & \forall t \in \mathbb{R}^+ - \mathscr{T}, \\ x(t^+) = A_r x(t), & \forall t \in \mathscr{T}, \\ x(0) = x_0 \in \mathbb{R}^n, & \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state of the system and $\mathscr{T} \in \Theta(\Delta)$, see [6]. By definition

$$t_{k+1} - t_k \in [\tau_m,\ \tau_M], \qquad \forall k \in \mathbb{N}$$

so we avoid Zeno phenomenon ($\tau_m > 0$) but an infinite number of reset instants occurs ($\tau_M < \infty$). The state at time $t \in (t_k,\ t_{k+1}]$, for a given initial state x_0 and a reset sequence $\mathscr{T} \in \Theta(\Delta)$ is given by

$$x(t) = e^{A_c(t-t_k)} A_r x(t_k), \qquad \forall t \in (t_k,\ t_{k+1}] \tag{3}$$

thus, the dynamics between two successive resets is given by the following discrete dynamics

$$x(t_{k+1}) = e^{A_c(t_{k+1}-t_k)} A_r x(t_k) = e^{A_c(\delta_k)} A_r x(t_k), \tag{4}$$

where $\delta_k = t_{k+1} - t_k \in \Delta$. Thus, denoting $A(\Delta) = \{e^{A_c\delta} A_r :\ \delta \in \Delta\}$, the problem of stability of the linear impulsive system (2) rewrites in terms of stability of the following discrete-time parametric uncertain system

$$\begin{cases} x^+ = A(\delta)x, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{5}$$

where $A(\delta) \in A(\Delta)$. Let us recall the definition of GES for the system (5).

Definition 1 *The system* (5) *is GES if there exist positive scalars $c \in \mathbb{R}$ and $\lambda \in [0,1)$ such that*

$$\|x(k)\| \leq c\lambda^k \|x_0\| \tag{6}$$

for every $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

It is noteworthy that systems (2) is GES if and only if (5) is GES. Notice that the set $A(\Delta)$ is not convex in general but it is compact, while the set in which the parameter δ lies, i.e. the interval Δ, is trivially convex and compact. Then, using the classical result from invariance and set-induced Lyapunov functions for linear (uncertain) discrete-time systems, see for instance [10, 11, 12], a necessary and sufficient condition for GES can be given, as follows.

Theorem 2 ([10, 11]) *There exists a Lyapunov function for a linear parametric uncertain system if and only if there exists a polyhedral Lyapunov function for the system.*

The theorem above is less conservative than Theorem 1 in [6], since it gives not only sufficient but also necessary condition for GES. It claims that the search of the candidate Lyapunov function can be limited to the family of functions which are induced by polytopes.

Remark 3 *It is noteworthy that the functions considered in Theorem 2 are convex, positive definite and homogeneous as in [6] (the fact that they are homogenous of order one and not of order two does not induce any loss of generality). Nevertheless, polyhedral Lyapunov functions are determined by a finite number of generators (the facets of the polytope they are induced by), then they form a set of functions strictly contained in the one considered in [6]. Therefore, the condition in Theorem 2 is less conservative and leads to necessary and sufficient conditions for stability which are computationally affordable, as shown in the sequel.*

We also recall another result, concerning set theory and its application to the problem of stability of linear uncertain systems. Given a C-set $\Omega \subseteq \mathbb{R}^n$, consider the following sequence of sets

$$\begin{cases} \Omega_0 &= \Omega, \\ \Omega_{k+1} &= Q_\lambda(\Omega_k, A(\Delta)) \cap \Omega, \end{cases} \tag{7}$$

where

$$Q_\lambda(S, \mathscr{A}) = \{x \in \mathbb{R}^n : \; Ax \in \lambda S, \; \forall A \in \mathscr{A}\} = \bigcap_{A \in \mathscr{A}} A^{-1}(\lambda S). \tag{8}$$

with $S \subseteq \mathbb{R}^n$ and $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$.

Lemma 4 *Given $\lambda \in \mathbb{R}^n$, $\Omega, \Gamma \subseteq \mathbb{R}^n$ and $\mathscr{A}, \mathscr{B} \subseteq \mathbb{R}^{n\times n}$ then*

a) $\mathscr{A} \subseteq \mathscr{B} \quad \Rightarrow \quad Q_\lambda(\Omega, \mathscr{A}) \supseteq Q_\lambda(\Omega, \mathscr{B})$.

b) $\Omega \subseteq \Gamma \quad \Rightarrow \quad Q_\lambda(\Omega, \mathscr{A}) \subseteq Q_\lambda(\Gamma, \mathscr{A})$.

c) If Ω is convex, then $Q_\lambda(\Omega, \mathscr{A}) = Q_\lambda(\Omega, co(\mathscr{A}))$.

d) If the Ω and $\mathscr{A}$ are polytopes, i.e.

$$\begin{aligned} &\Omega = \{x \in \mathbb{R}^n : Hx \leq b\}, \\ &\mathscr{A} = \{A \in \mathbb{R}^{n\times n} : \; A = \sum_{i \in \mathbb{N}_a} \alpha_i A_i, \, \alpha_i \geq 0, \, \sum_{i \in \mathbb{N}_a} \alpha_i = 1\}, \end{aligned}$$

then $Q_\lambda(\Omega, \mathscr{A})$ is a polytope too.

Proof: The proof requires just some careful but straightforward mathematical manipulations which are not presented here. A journal version containing detailed proof is in preparation. ■

Definition 5 *For any $\lambda \in [0,1)$ we say the set S is λ-contractive w.r.t. dynamics* (5) *if and only if*

$$A(\delta)x \in \lambda S, \quad \forall x \in S, \; \delta \in \Delta.$$

As proven in [13], the maximal λ-contractive set w.r.t. dynamics (5), which is contained in Ω, is given by

$$\Omega_\lambda = \bigcap_{k \in \mathbb{N}} \Omega_k, \tag{9}$$

where Ω_k are defined by (7). We note that, due to the linearity of (5), Ω_λ is compact and convex as far as Ω is a C-set. However, Ω_λ is not always a C-set since, for some values of λ, it can be reduced to the origin. In the following we denote λ^* the infimum in $[0,1)$ for which Ω_{λ^*} is a C-set.

Theorem 6 (Theorem 3.2 in [13]) *For $\lambda \in [0, \; 1)$ let us assume that Ω_λ defined by (7) and (9) is a C-set. Then, for every $\mu \in (\lambda, 1]$ there exists j such that Ω_k is μ-contractive for all $k \geq j$.*

Remark 7 *From Theorem 6, it follows that, for any $\mu > \lambda^*$ we can obtain a μ-contractive C-set w.r.t. dynamics* (5) *by iterating* (7) *a finite number of times with $\lambda \in [\lambda^*, \mu)$.*

Therefore, the iteration (7) together with an appropriate stop condition, represents one version of the basic algorithm for obtaining a λ-contractive set w.r.t. (5). Moreover, the algorithm terminates in a finite number of steps provided λ is adequately chosen.

Definition 8 *Given a C-set $S \subseteq \mathbb{R}^n$, its Minkowski functional $\Psi_S : \mathbb{R}^n \to \mathbb{R}$ is defined as*

$$\Psi_S(x) = \min_{\alpha \geq 0}\{\alpha \in \mathbb{R} : \ x \in \alpha S\}. \tag{10}$$

Proposition 9 *The linear parametric uncertain system (5) is GES if and only if for every C-set Ω there exists $\lambda \in [0,\ 1)$ such that for all $\mu \in (\lambda, 1)$ there is $k = k(\lambda, \mu) \in \mathbb{N}$ such that*

$$\Omega_k \subseteq Q_\mu(\Omega_k, A(\Delta)), \tag{11}$$

with Ω_k as in (7). Moreover, $\Psi_{\Omega_k}(x)$ is a global exponential Lyapunov function for (5).

Proof: The result comes directly from Theorem 6 and the fact that the condition (11) is equivalent to μ-contractivity of the set Ω_k (see [11, 12]). Indeed, (11) is equivalent to the fact that for all $x \in \Omega_k$, x belongs also to $Q_\mu(\Omega_k, A(\Delta))$ which means, by definition (8), that $A(\delta)x \in \mu\Omega_k$ for every $\delta \in \Delta$, definition of μ-contractivity of Ω_k.
■

Alternative, but analogous, formulations of the stop conditions are given in literature, see [12]. Thus, summarizing, classical literature results on invariance and set-induced Lyapunov functions permit to assert that the class of positive definite polyhedral Lyapunov functions, that is, the Minkowski functions of polytopic C-sets, forms a universal class of Lyapunov functions for assessing GES for parametric uncertain linear systems. Moreover, algorithms exist such that contractive sets (and then also the related set-induced Lyapunov function) can be obtained after a finite number of iterations for exponentially stable parametric uncertain systems.

Problem 10 *Given an exponentially stable uncertain system (5), an initial polytopic C-set and a λ such that a C-set λ-contractive exists, does the recursion (7) with stop condition (11) provide a λ-contractive polytope?*

The answer depends on the assumptions on $A(\Delta)$. It has been proven that if $A(\Delta)$ is a polytope, then the algorithm provides λ-contractive polytopes [13, 12]. Such results follow directly from the fact that $Q_\lambda(\cdot, A(\Delta))$ maps polytopes into polytopes provided that $A(\Delta)$ is a polytope in $\mathbb{R}^{n\times n}$. Nevertheless, supposing that $A(\Delta)$ is just a compact set, such property is no more ensured in general.

Example 11 *Consider the discrete-time linear uncertain system (5) with $A(\delta) = \alpha R(\delta)$ where $R(\delta)$ is the rotation matrix, i.e.*

$$R(\delta) = \begin{bmatrix} \cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta) \end{bmatrix}, \tag{12}$$

with $\delta \in \Delta = [0,\ \pi/4]$ and $\alpha \in (0,1)$, which ensures robust asymptotic stability. The set $A(\Delta)$ is not a polytope, neither a convex set, in $\mathbb{R}^{2\times 2}$. Notice, the $A(\delta)$ is related to a contraction and turn dynamics. Given a set Ω, the set of successor and predecessor states of $x \in \Omega$ for the system (5) are

$$A(\Delta)\Omega = \bigcup_{\delta\in\Delta} A(\delta)\Omega = \bigcup_{\delta\in\Delta} \alpha R(\delta)\Omega = \{x \in \mathbb{R}^2 : \ x = \alpha R(\delta)z, \ \ z \in \Omega, \forall \delta \in \Delta\},$$

$$
\begin{aligned}
A(\Delta)^{-1}\Omega &= \bigcap_{\delta\in\Delta} A(\delta)^{-1}\Omega = \bigcap_{\delta\in\Delta} \alpha^{-1}R(-\delta)\Omega \\
&= \bigcap_{\delta\in\Delta} \{x\in\mathbb{R}^2:\ x=\alpha^{-1}R(-\delta)z,\ \ z\in\Omega\} \\
&= \bigcap_{\delta\in\Delta} \{x\in\mathbb{R}^2:\ \alpha R(\delta)x\in\Omega\} \\
&= \{x\in\mathbb{R}^2: \alpha R(\delta)x\in\Omega, \forall\delta\in\Delta\} = Q_1(\Omega, A(\Delta)).
\end{aligned}
$$

Geometrically it means that, for every $\Omega\subseteq\mathbb{R}^n$, the set $Q_\lambda(\Omega,A(\Delta))$ is given by the intersection of $\alpha^{-1}\Omega$ rotated by $-\delta$, for every $\delta\in[0,\ \pi/4]$. Therefore the set $Q_\lambda(\Omega,A(\Delta))$ is not, in general, a polytope, neither for polytopic Ω. Then there is no insurance that the λ-invariant set potentially provided by the recursion (7) is a polytope. In fact, we have that

$$
\Omega_{k+1} = \{x\in\Omega_0:\ \alpha R(\delta)x\in\lambda\Omega_k,\ \ \forall\delta\in\Delta\} = \{x\in\Omega_0:\ R(\delta)x\in\alpha^{-1}\lambda\Omega_k,\ \ \forall\delta\in\Delta\},
$$

which is given by the intersection of an infinite number of sets, one for every $\delta\in\Delta$. Consider, for instance the case of $\alpha=0.9$ and apply the recursion with $\Omega_0=\mathbf{B}^2_\infty$ and $\lambda=0.9$. We obtain, at the first step, Ω_1 depicted in Figure 1 (left), non-polytopic.

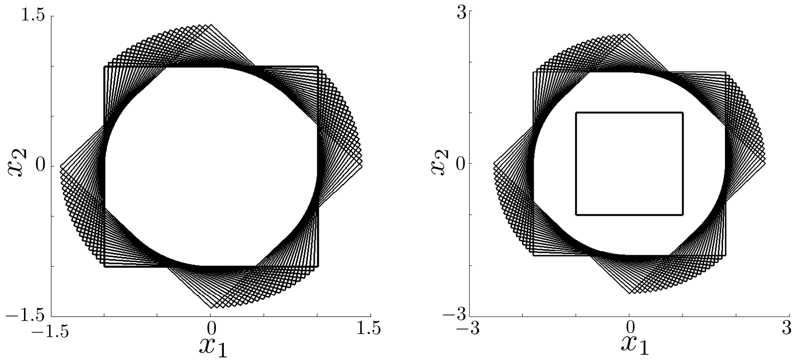

Figure 1: Ω_0 and $Q_\lambda(\Omega,A(\Delta))$ for $\alpha=0.9$ (left) and $\alpha=0.5$ (right).

Nevertheless, if we choose α small enough for the contraction to compensate the rotation due to the uncertainty, then a λ-contractive polytope is obtained at the first step. Figure 1 at right represents Ω_0 and $Q_\lambda(\Omega,A(\Delta))$ for $\alpha=0.5$. It is obvious that $\Omega_1=\Omega_0$ and that it is a λ-contractive polytope.

Our main objective is to provide a variation of the classical recursive algorithm for contractive sets computation, such that a polytopic contractive set, and then a polyhedral set-induced Lyapunov functions, can be obtained in finite time. Moreover, such algorithm should have a computional complexity analogous to the classical one. The algorithm is afterward adapted to the case of study of nearly-periodic reset systems.

0.3 Minkowski functional formalism

In this section we present more details on the Minkowski functional, which is main tool used in the sequel to obtain necessary and sufficient condition for GES of (5). Due to space limitations we do not provide the proofs of the instrumental lemmas in this section. A journal version containing detailed proof is in preparation.

Definition 12 *Given a C-set $\Omega \subseteq \mathbb{R}^n$, define:*

- *Minkowski functional of a compact set $S \subseteq \mathbb{R}^{n\times n}$:*

$$\Psi_\Omega(S) = \max_{x\in S} \Psi_\Omega(x),$$

- *Minkowski functional of a matrix $A \in \mathbb{R}^{n\times n}$, as induced by the functional for a vector:*

$$\Psi_\Omega(A) = \max_{x\in \Omega} \Psi_\Omega(Ax).$$

- *Minkowski functional of compact sets of matrices $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$, as induced by the functional for a matrix:*

$$\Psi_\Omega(\mathscr{A}) = \max_{A\in \mathscr{A}} \Psi_\Omega(A).$$

Notice: if Ω is a symmetric C-set, then $\Psi_\Omega(x)$ is a vector norm ([14, 12]).

Definition 13 *The (Hausdorff) distance induced by the Minkowski functional of the C-set $\Gamma \subseteq \mathbb{R}^n$ in the space of matrices $\mathbb{R}^{n\times n}$ is defined*

$$d_\Gamma(\mathscr{A},\mathscr{B}) \triangleq \inf\{\alpha \geq 0 : \mathscr{A} \subseteq \mathscr{B} + \alpha \mathbf{B}_\Gamma^{n\times n}; \mathscr{B} \subseteq \mathscr{A} + \alpha \mathbf{B}_\Gamma^{n\times n}\}$$

where

$$\begin{aligned}\mathbf{B}_\Gamma^{n\times n} &\triangleq \{A \in \mathbb{R}^{n\times n} : \Psi_\Gamma(A) \leq 1\}\\ &= \{A \in \mathbb{R}^{n\times n} : \Psi_\Gamma(Ax) \leq \Psi_\Gamma(x)\}.\end{aligned}$$

Lemma 14 *If Ω is a symmetric C-set, then $\Psi_\Omega(x)$ is a vector norm and $\Psi_\Omega(A)$ is the induced operator norm.*

Remark 15 *Given the C-set $\Omega \subseteq \mathbb{R}^n$, one has*

$$\Psi_\Omega(\mathscr{A}x) \leq \Psi_\Omega(\mathscr{A})\Psi_\Omega(x)$$

The next lemma follows from convexity of Ω.

Lemma 16 *Given the C-set $\Omega \subseteq \mathbb{R}^n$, $\Psi_\Omega(\mathscr{A})$ is such that*

$$\Psi_\Omega(\mathscr{A}) = \Psi_\Omega(co(\mathscr{A})). \tag{13}$$

Remark 17 *Given the C-set $\Omega \subseteq \mathbb{R}^n$ and $\mathscr{A},\mathscr{B} \subseteq \mathbb{R}^{n\times n}$ such that $\mathscr{A} \subseteq co(\mathscr{B})$ then $\Psi_\Omega(\mathscr{A}) \leq \Psi_\Omega(\mathscr{B})$. The inverse implication is not true in general: consider for instance $\mathscr{A} = \{0\}$ and $\mathscr{B} \neq \{0\}$. Then $\Psi_\Omega(\mathscr{A}) = 0 < \Psi_\Omega(\mathscr{B})$ but $\mathscr{A} \nsubseteq co(\mathscr{B})$.*

0.4 Stability analysis for linear impulsive systems

In this section we provide tractable necessary and sufficient condition for stability of (5) and present the algorithm that allows computing the associated polyhedral Lyapunov function.

0.4.1 Tractable necessary and sufficient condition

First we provide a necessary condition, together with its implication, for a set to be λ-contractive for the linear uncertain system (5).

Proposition 18 *If the linear parametric uncertain system (5) is GES then for every C-set Ω and for all $\mathscr{A} \subseteq co(A(\Delta))$ there exists $\lambda \in [0,\ 1)$ such that for all $\mu \in (\lambda, 1)$ there is $p = p(\lambda, \mu) \in \mathbb{N}$ such that condition*

$$\Omega_k \subseteq Q_\mu(\Omega_k, \mathscr{A}), \qquad \forall k \geq p, \tag{14}$$

holds, with Ω_k given by

$$\begin{cases} \Omega_0 &= \Omega, \\ \Omega_{i+1} &= Q_\lambda(\Omega_i, \mathscr{A}) \cap \Omega. \end{cases} \tag{15}$$

Moreover, if Ω is a polytope in $\mathbb{R}^n$ and $co(\mathscr{A})$ a polytope in $\mathbb{R}^{n\times n}$ then Ω_k are polytopes and $\Psi_{\Omega_k}(x)$ is a polyhedral global exponential Lyapunov function for the system $x^+ \in \mathscr{A}x$.

Proof: The result follows directly from Theorem 6 and the fact that if $x^+ \in A(\Delta)x$ is GES, also $x^+ \in \mathscr{A}x$ is GES. ■

The Proposition 18 substantially claims that if one replaces the uncertainty set $A(\Delta)$ with a set which is either polytopic or finite and contained in $\mathrm{co}(A(\Delta))$, then the recursion generates sequences of polytopes and terminates with a polytopic contractive set, if the system is exponentially stable. Notice that this entails a relaxion of the uncertainty bounds and then to an only necessary condition. On the other hand, this leads to a first computationally tractable recursion for obtaining approximation of the polytopic contractive set for (5).

Corollary 19 *Given $\Omega \subseteq \mathbb{R}^n$ polytope with $0 \in int\ (\Omega)$ and $\mathscr{A} = \{A_i\}_{i=1}^N \subseteq co(A(\Delta))$, then the recursion (15) with stop condition (14) terminates in finite steps for appropriate values of $\lambda \in [0,\ 1)$ and $\mu \in (\lambda,\ 1)$ if the system (5) is GES.*

Then, provided the system is GES, every finite selection of matrices in $\mathrm{co}(A(\Delta))$ gives in finite time a polytopic contractive set and a polyhedral Lyapunov function, for adequate λ and μ. This also means that, if one proves that no contractive set exists for an uncertain system whose matrices forms a subset of $\mathrm{co}(A(\Delta))$, then the system is not exponentially stable.

Corollary 20 *Given $\Omega \subseteq \mathbb{R}^n$ polytope with $0 \in int\ (\Omega)$ and $\mathscr{A} = \{A_i\}_{i=1}^N \subseteq co(A(\Delta))$, if there are not $\lambda \in [0,\ 1)$ and $\mu \in (\lambda,\ 1)$ such that the stop condition (11) holds for recursion (15), then the system (5) is not GES.*

The main practical drawback of the latter result is that, in general, it is not trivial to prove that no such pair of λ and μ exists.

Let us consider an increasing sequence of inner approximations of the set $\mathrm{co}(A(\Delta))$ (for everyone of which a contractive set exists, from Corollary 19) that converges to $\mathrm{co}(A(\Delta))$. Let us also consider the corresponding sequence of contractive sets obtained by means of (14) and (15). The main idea is to prove that the latter sequence converges to a polytopic contractive set for system (5), if and only (5) is GES.

Remark 21 *The metric space of the compact sets of $\mathbb{R}^{n\times n}$ equipped with the Hausdorff distance (determined by the unitary ball with respect to a matricial induced norm) is complete, see [15, 16].*

Theorem 22 *The linear parametric uncertain system (5) is GES if and only if for every C-set Ω and every increasing sequence of compact convex sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ such that $\mathscr{A}^{(j)} \subseteq co(A(\Delta))$ and*

$$\lim_{j\to\infty} co(\mathscr{A}^{(j)}) = co(A(\Delta)), \tag{16}$$

there exists $\lambda \in [0,1)$, $\nu \in (\lambda, 1)$, $k = k(\lambda, \nu) \in \mathbb{N}$ and $h = h(\lambda, \nu) \in \mathbb{N}$ such that condition

$$\Omega_k^{(h)} \subseteq Q_\nu(\Omega_k^{(h)}, A(\Delta)) \tag{17}$$

holds, with the sequence of sets $\Omega_k^{(j)}$ given by

$$\begin{cases} \Omega_0^{(j)} &= \Omega, \\ \Omega_{i+1}^{(j)} &= Q_\lambda(\Omega_i^{(j)}, \mathscr{A}^{(j)}) \cap \Omega. \end{cases} \tag{18}$$

Moreover, if Ω is a polytope in $\mathbb{R}^n$ and $co(\mathscr{A}^{(j)})$ are polytopes in $\mathbb{R}^{n\times n}$ then $\Omega_k^{(j)}$ are polytopes and $\Psi_{\Omega_k^{(h)}}(x)$ is a polyhedral global exponential Lyapunov function for (5).

Remark 23 *Notice that λ and k do not necessarily depend on $\mathscr{A}^{(j)}$, whereas ν and h do. Moreover, from the practical point of view, it is worth noting that the value of μ, ρ and h don't have to be computed. Theorem 22 claims that, by choosing appropriate $\lambda \in [0,1)$ and $\nu \in (\lambda,1)$, the sets $\Omega_k^{(j)}$ are ν-contractive for all j big enough. Thus, the computational complexity is analogous to that of classical algorithm for contractive sets computation. The shape of the computed contractive sets could be complex, but this is related to the complexity of the problem itself.*

Thus, any sequence of compact sets $\mathscr{A}^{(j)}$ whose convex hull converges from the interior to the convex hull of $A(\Delta)$ generates a sequence of C-sets $\Omega_k^{(j)}$ that converges to a contractive set for (5). Remarkably, if the sets $\mathscr{A}^{(j)}$ are polytopes or finite sets (and Ω is a polytope), the sets $\Omega_k^{(j)}$ are also polytopes.

Corollary 24 *Let the linear parametric uncertain system (5) be GES and consider $\lambda \in [0,1)$, $\mu \in (\lambda,1)$, $k = k(\lambda,\mu) \in \mathbb{N}$ such that Ω_k is μ-contractive. Then, for every $\nu \in (\mu,1)$ and every increasing sequence of compact convex sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ such that $\mathscr{A}^{(j)} \subseteq co(A(\Delta)$ with (16) there exists $h = h(\lambda,\nu)$ such that $\Omega_k^{(j)}$ given by (18) is ν-contractive for (5) for all $j \geq h$.*

0.4.2 Computation of contractive polytopes and polyhedral Lyapunov functions

The basic idea for certifying a nearly periodic reset system is GES, is to generate appropriate inner approximations of the set $A(\Delta)$ and use it to compute a contractive C-set. Since every sequence $\mathscr{A}^{(j)}$ whose convex hull converges to the one of $A(\Delta)$ eventually lead to a contractive C-set for (5), we can restrict our attention to finite sets $\mathscr{A}^{(j)}$. This, together with polytopic Ω would lead to sequences of polytopic $\Omega_k^{(j)}$, thus numerically suitable.

Remark 25 *An important computational implication of considering inner approximations of $co(A(\Delta))$ rather than outer ones, as for instance in [6], is that they are much easier to be obtained. In fact, every finite set $\mathscr{A}$ contained in $co(A(\Delta))$ is an inner approximation. Moreover, adding a matrix $A \notin \mathscr{A}$ such that $A \in co(A(\Delta))$ to $\mathscr{A}$ leads to a tighter approximation of $co(A(\Delta))$. Then, the sequences $\mathscr{A}^{(j)}$ can be easily genarated by adequately selecting points on the boundary of $co(A(\Delta))$. Hence no relevant computational effort is required to generate the sequence $\mathscr{A}^{(j)}$.*

Thus, generating an appropriate sequence $\mathscr{A}^{(j)}$ with $j \in \mathbb{N}$ such that (16) is satisfied is a tractable problem in general, even for non-polytopic and nonconvex sets $A(\Delta)$. Then, the only main computational issue for the practical application of the result of Theorem 22 is checking whether the condition (17) is satisfied, i.e. if

$$\begin{aligned} &Ax \in \nu\Omega_k^{(j)}, \quad \forall A \in A(\Delta),\, x \in \Omega_k^{(j)} \Leftrightarrow \\ &A\Omega_k^{(j)} \subseteq \nu\Omega_k^{(j)}, \quad \forall A \in A(\Delta). \end{aligned}$$

Indeed, if the set $A(\Delta)$ is not polytopic (in which case a finite number of matrices $A \in A(\Delta)$ should suffice to be checked), condition (17) concerns an uncountable number of matrices in $A(\Delta)$. A possible approach could consist in evaluating the condition for an outer polytopic set $\mathscr{A}$, i.e. for $A \in \mathscr{A}$ with $\mathscr{A}$ polytopic and $\mathrm{co}(A(\Delta)) \subseteq \mathscr{A}$. Nevertheless, once more, the computation of outer approximations of $\mathrm{co}(A(\Delta))$ could be numerically inefficient, besides of introducing a certain conservatism.

The following considerations are aimed at providing tractable conditions to check whether (17) is satisfied.

Given the two generic sets $\Lambda \subseteq \mathbb{R}^{p\times n}$ and $\mathscr{A} \subseteq \mathbb{R}^{n\times m}$ define

$$\Lambda\mathscr{A} = \bigcup_{\Gamma\in\Lambda} \Gamma\mathscr{A} = \bigcup_{\Gamma\in\Lambda}\bigcup_{\Sigma\in\mathscr{A}} \Gamma\Sigma.$$

Proposition 26 *Suppose that $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ compact is such that for every C-set Ω there exists $\lambda \in [0,\ 1)$ such that for all $\mu \in (\lambda, 1)$ there is $k = k(\lambda,\mu) \in \mathbb{N}$ such that condition (14) holds, with Ω_k given by the sequence of sets given by (15). If $\Lambda \subseteq \mathbb{R}^{n\times n}$ is such that*

$$A(\Delta) \subseteq \Lambda\, co(\mathscr{A}), \tag{19}$$

with

$$\Psi_{\Omega_k}(\Lambda) < \mu^{-1}, \tag{20}$$

then the linear parametric uncertain system (5) is GES and $\Psi_{\Omega_k}(x)$ is a global exponential Lyapunov function for (5).

Theorem 27 *The system (5) is GES if and only if for every two sequences of compact sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$, increasing, and $\{\Lambda^{(j)}\}_{j\in\mathbb{N}}$ such that*

$$\mathscr{A}^{(j)} \subseteq co(A(\Delta)) \subseteq \Lambda^{(j)} co(\mathscr{A}^{(j)}), \tag{21}$$

and

$$\lim_{j\to+\infty} \Lambda^{(j)} = I, \tag{22}$$

there exist $\lambda \in [0,1)$, $\nu \in (\lambda,1)$, $k \in \mathbb{N}$ and $h \in \mathbb{N}$ such that $\Omega_k^{(h)}$, given by (18) is ν-contractive for $x^+ \in \mathscr{A}^{(h)}x$ and

$$\Psi_{\Omega_k^{(h)}}(\Lambda^{(j)}) < \nu^{-1}. \tag{23}$$

0.4.3 Finitely determined polytopic Lyapunov functions

We sketch here the procedure for obtaining polyhedral exponential Lyapunov functions, and thus for checking if the nearly-periodic reset system is GES. Many important computational issues, that would deserve to be deeply analysed, are the objective of our current and future research.

The first step concern a possible method to generate the sequence of sets in the space of matrices $\Lambda^{(j)}$ and $\mathscr{A}^{(j)}$, with $j \in \mathbb{N}$ such that conditions (21) and (22) hold. Precisely, the sequences of sets $\Lambda^{(j)}$ and $\mathscr{A}^{(j)}$, with $j \in \mathbb{N}$ satisfying (21) and (22) can be constructed as follows:

$$\begin{aligned} &\mathscr{A}^{(j)} = \bigcup_{i=0}^{j} e^{A_c \tau^{(j)} i} A_m, \quad \tau^{(j)} = (\tau_M - \tau_m) j^{-1}, \\ &\Delta^{(j)} = [0, \tau^{(j)}], \quad \Lambda^{(j)} = \bigcup_{\delta \in \Delta^{(j)}} e^{A_c \delta}, \end{aligned} \tag{24}$$

with $A_m = e^{A_c \tau_m} A_r$.

Then, the testing procedure consists in iterating through $j \in \mathbb{N}$ to obtain the structures in (24). For every j, a polytope $\Omega_k^{(j)}$ that is robustly contractive for $\mathscr{A}^{(j)}$ is computed (where k depends on j) and then condition (23) is checked. If it holds, then the polytope $\Omega_k^{(j)}$ is contractive also in presence of the uncertainty induced by $\Lambda^{(j)}$ and thus the impulsive system is GES and the function induced by $\Omega_k^{(j)}$ is a Lyapunov function.

0.5 Numerical example

Consider the impulsive system (2) with matrices

$$A_r = \begin{bmatrix} 0.5 & -0.25 \\ 0.5 & 1 \end{bmatrix}, \quad A_c = \begin{bmatrix} 0.1 & -1 \\ 1 & 0.1 \end{bmatrix} \tag{25}$$

and $\Delta = [0.5, 1.5]$. Notice that, whereas the discrete-time transition matrix A_r is Schur, A_c has two complex conjugate poles with positive real part and then

$$e^{A_c t} = \begin{bmatrix} e^{0.1t}\cos(t) & -e^{0.1t}\sin(t) \\ e^{0.1t}\sin(t) & e^{0.1t}\cos(t) \end{bmatrix},$$

thus, the continuous-time trajectories are diverging spirals. Such divergence must be compensated by the reset action to have stability. Notice that $A(\Delta)$ is neither a polytope nor a convex set. Applying the procedure illustrated in Section 0.4.3 we proved that the system is GES. For different values of j we computed $\Omega_k^{(j)}$ for an appropriate λ and then check if (23) holds (with, in our case, $\lambda = \nu$). We found that for different values of $j > 1$, the condition can be satisfied whereas is not for $j = 1$. Some numerical results are summarized in Table 1. Thus, the impulsive system with (25) is GES. Although the analysis with $j = 2$ would have been sufficient to asses GES, we wanted to stress that the procedure can be applied for much higher j highlighting the computational flexibility of the approach.

Table 1:

	$j=1$	$j=2$	$j=5$	$j=10$	$j=12$	$j=15$
λ	0.874	0.88	0.90	0.91	0.91	0.92
k	4	10	9	9	9	9
$\Psi_{\Omega_k^{(j)}}$	1.1554	1.1149	1.0234	1.0128	1.0175	1.0190
μ^{-1}	1.1442	1.1364	1.1111	1.0989	1.0989	1.0870

0.6 Conclusions

In this paper we employ set theory to provide a tractable method for testing whether an impulsive linear system is globally exponentially stable. The reset rule considered in this paper is assumed to be nearly-periodic. We provide a method for obtaining a polyhedral Lyapunov function, whose existence is necessary and sufficient for the system to be GES. The approach is particularly suitable since the computational burden is analogous to that required for linear uncertain polytopic systems. Many issues related to the computational aspects are among the objectives of our current research.

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